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Department of Information and Computer Science

Question	1	2	3	4	5	Total
Max	6	6	6	6	6	30
Earned						

Question 1: [6 Points] **[Structural Induction]** [CLO #2] Use structural induction to show that if T is a full binary tree, then $l(T) = i(T) + 1$. Where $l(T)$ is the number of leaves of T and $i(T)$ is the number of internal vertices of T. Note: Leaves are nodes with no children.

Base Step:
(T = leaf).
We have $l(\text{leaf}) = 1 = 1 + 0 = 1 + i(\text{leaf})$
Induction step:
Assume $l(T_1) = 1 + i(T_1)$ and $l(T_2) = 1 + i(T_2)$, where T_1 and T_2 are full binary trees and ($T = T_1 \cdot T_2$).
We have

$l(T_1 \cdot T_2) = l(T_1) + l(T_2)$	by the def. of l
$= 1 + i(T_1) + l(T_2)$	by the IH for T_1
$= 1 + i(T_1) + 1 + i(T_2)$	by the IH for T_2
$= 1 + i(T_1 \cdot T_2)$	by the def. of i

We have completed all the cases for structural induction on a binary tree, so we can therefore conclude that for any full binary tree T, $l(T) = 1 + i(T)$.

Question 2: [6 Points] **[Strong Induction]** [CLO #2] Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. Using Strong induction, prove that $P(n)$ is true for $n \geq 8$.

Base step: We need to show that $P(8)$, $P(9)$, and $P(10)$ are all true.
We can write $8 = 3 + 5$ $9 = 3 + 3 + 3$ $10 = 5 + 5$,
so the statements $P(8)$, $P(9)$, and $P(10)$ are all true.

Induction Step: since we'll be using strong induction, we assume that "$P(n)$ is true for all n with $8 \leq n \leq k$, where $k \geq 10$ is some integer."
We need to prove that, assuming $P(k)$ we $P(k + 1)$ is true for $k \geq 10$; that is, that $k + 1$ can be written as a sum of 3's and 5's.

We wish to write $k + 1$ as a sum of 3's and 5's. BY THE INDUCTIVE HYPOTHESIS, P(k – 2) is true (this is why we had to cover 8, 9, and 10 as base cases), letting us write
$k - 2 = 3s + 5r$.
Adding 3 to both sides gives us
$k + 1 = 3s + 5r + 3 = 3(s + 1) + 5r$,
so the $P(k + 1)$ is true.

Since the base cases $n = 8$, $n = 9$, and $n = 10$ all held, and since the inductive step held for $k \geq 10$, $P(n)$ holds for all $n \geq 8$. That is, if n is an integer ≥ 8, we can make a postage of n cents just using 3-cent stamps and 5-cent stamps.

Question 3: [6 Points] **[Counting & Probability]** [CLO #3] Suppose that 210 people enter a contest and that different winners are selected at random for first, second, third, fourth, and fifth prizes. What is the probability that Ali, Salem, Ameer, Waleed, and Ahmad each win a prize if each has entered the contest? Show your answer using combination and division of numbers.

Any group of 5 people is equally likely to win the 5 prizes.

There are $\binom{100}{5}$ groups of 5 people,

so the probability is $\frac{1}{\binom{100}{5}}$

Question 4: [6 Points] **[Permutations & Combinations]** [CLO #3] How many bit strings of length 16 with exactly two 0's are there for which the 0's are not adjacent?

As we have exactly two 0's we have exactly fourteen ones. We write the fourteen ones with spaces between them to count the number of possible places where 0 can be inserted.

1 1 1 1 1 1 1 1 1 1 1 1 1 1

There are thirteen places between the 1's, one place before the string, and one place after the string- a total of fifteen not-adjacent possible places. We want to select two places out of these fifteen places. So the answer is $C(15,2) = \binom{15}{2} = \frac{15!}{13!2!} = 105$

Question 5: [6 Points] **[Recursive Definition]** [CLO #3] A bit string that reads the same backwards as forwards is called a palindrome. That is a palindrome is a string which, when reversed it is identical to the original string, eg. 0110 is a palindrome and 0100 is not.

More examples of bit string palindromes are: 0, 1, 00, 11, 010, 101, 000, 111, 1111, 10101, 01010, ...

Give a recursive definition of the set of bit strings that are palindrome.

Basis Step:
$\lambda \in S$ (λ is the empty string)
$0 \in S$
$1 \in S$
Recursive Step: If $x \in S$, then $0x0 \in S$ and $1x1 \in S$